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Estimation of photon counting statistics with imperfect detectors*

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The study on photon counting statistics is of fundamental importance in quantum optics. We theoretically analyzed the imperfect detection of an arbitrary quantum state. We derived photon counting formulae for six typical quantum states (i.e., Fock, coherent, squeeze-vacuum, thermal, odd and even coherent states) with finite quantum efficiencies and dark counts based on multiple on/off detector arrays. We applied the formulae to the simulation of multiphoton number detections and obtained both the simulated and ideal photon number distributions of each state. A comparison between the results by using the fidelity and relative entropy was carried out to evaluate the detection scheme and help select detectors for different quantum states.

Keywords: fidelity, on/off detector, photon number detection, relative entropy

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1. Introduction

Quantum optics is recognized as a counter-intuitive theory because of its non-classical properties. It has been widely used in applications such as quantum information technology,[1–5] quantum teleportation,[6–8] and quantum computation.[9–12] Photon number detection[13,14] is considered as an essential branch of quantum technology in photonic systems. The unknown quantum state to be measured does not always consist of only one photon. For multiphoton detections, photon-number-resolving (PNR) detectors[15–17] are regarded as the simplest apparatus as the photon number distribution can be provided directly. However, they are not widely used because of their low efficiency.[18] The on/off detectors made of avalanche photodiodes exhibit a high efficiency,[19,20] but the number of incoming photons cannot be distinguished. Sperling *et al.* proposed a robust method for photon detection by using an on/off detector array instead of the PNR detector.[21] The schematic is illustrated in Fig. 1, where an unknown quantum state is divided equally into N ports via a series of hypothetical unitary operations. At each port, there is an on/off detector for photon detection. The total number of clicks of the detectors can then be counted, which is approximately recognized as the photon number distribution. The unitary operations here are assumed to be ideal.

As it is possible that more than one photon can enter the same port in this method, the results may not be acceptable when there are insufficient on/off detectors. However, when there are sufficient on/off detectors, the probability of this situation is lower, and the detection results can then approach those of the ideal case. Here, we proposed a method using the fidelity and relative entropy to characterize the performance of the detectors. The photon counting formulae could be deduced based on multiple on/off detector arrays considering both the quantum efficiency and dark counts. Using these formulae, we took six typical quantum states as examples to study the number of detectors that can be used to obtain an ideal result as well as the effects arising from the noise counts.

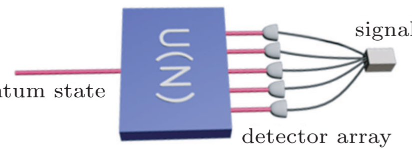

Fig. 1. (color online) Schematic of multiphoton detections based on an on/off detector array. An unknown quantum state is equally divided into N ports by a series of hypothetical unitary operations ($U(N)$). At the end of each port, an on/off detector is ready for the detection of a possible incoming photon. The photon number distribution is obtained by summing the clicks of all the on/off detectors.

2. Photon counting formulae based on on-off detectors

In the method based on the on/off detector arrays, the photon-counting formula of the quantum state $|\psi\rangle$ can be written as[22]

$$p_k(\psi) = \mathrm{Tr}[\hat{\rho}_\psi \hat{\Pi}_k(N)], \tag{1}$$

where $\hat{\rho}_\psi$ is the density operator of the quantum state $|\psi\rangle$, N is the number of on/off detectors, and

$$\begin{aligned}\hat{\Pi}_k(N) =&: C_N^k \exp[-(\eta\hat{n}+N\upsilon)/N]^{(N-k)} \\ &\times\{I-\exp[-(\eta\hat{n}+N\upsilon)/N]\}^k :\end{aligned} \tag{2}$$

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is the projection operator.[21] Here, η is the quantum efficiency, υ is the dark counts, and : : is the normal ordering prescription. By combining Eqs. (1) and (2), we can obtain the detection probability when k detectors click (see Appendix A)

$$p_k(\psi) = \sum_{l=0}^{\infty} Q_l(\psi,\eta) M_l(N,k,v), \tag{3}$$

where

$$Q_l(\psi,\eta) = \sum_{n=0}^{\infty} C_n^l (1-\eta)^{n-l} \eta^l \langle n|\rho_\psi|n\rangle, \tag{3a}$$

$$M_l(N,k,v) = \sum_{m=0}^{\infty} \frac{v^m}{m!} \cdot \exp(-Nv) N^{-l} S(m+l,k) P_N^k. \tag{3b}$$

Here, $S(m+l,k)$ is the Stirling number of the second kind,[23] C_{l+n}^l is the binomial coefficient, and $P_N^k = N(N-1)\cdots(N-k+1)$ is the permutation coefficient. As only the term $Q_l(\psi,\eta)$ contains the information of quantum states and $M_l(N,k,\upsilon)$ is only the normalization coefficients, we can write $Q_l(\psi,\eta)$ in the form of different quantum states depending on their unique expressions as follows.

For the Fock state $|n\rangle$,

$$Q_l(\psi,\eta) = C_n^l (1-\eta)^{n-l} \eta^l. \tag{4}$$

For the thermal state,

$$\rho_{\text{th}} = \sum_n (1-\mathrm{e}^{-x})\,\mathrm{e}^{-nx} |n\rangle\langle n|,$$

$$Q_l(\psi,\eta) = \frac{(\bar{n}\eta)^l}{(1+\bar{n}\eta)^{l+1}}. \tag{5}$$

For the squeezed-vacuum state,

$$|\xi\rangle = \frac{1}{\sqrt{\cosh\xi}} \sum_{m=0}^{\infty} \left(\frac{\tanh\xi}{2}\right)^m \frac{\sqrt{(2m)!}}{m!} |2m\rangle,$$

$$Q_l(\psi,\eta) = A P_l(\mathrm{i}x)(-\mathrm{i}y)^l, \tag{6}$$

where $P_l(z)$ is the Legendre function,[24] $A = [\cosh^2\xi - (1-\eta)^2 \sinh^2\xi]^{-1/2}$, $x = A(1-\eta)\sinh\xi$, and $y = A\eta\sinh\xi$ (see Appendix B). For a coherent state $|\alpha\rangle = \exp(\alpha a^+ - \alpha^* a)|0\rangle$, the photon counting probability p_k can be written as

$$p_k(\alpha) = C_N^k \exp[-(\eta|\alpha|^2 + Nv)/N]^{(N-k)} \times \{I - \exp[-(\eta|\alpha|^2 + Nv)/N]\}^k. \tag{7}$$

For a coherent superposition state $|\psi\rangle = M(|\alpha\rangle + \exp(\mathrm{i}\varphi)|-\alpha\rangle)$, where $|\alpha\rangle$ is a coherent state, the photon counting probability p_k can be written as

$$p_k(\psi) = 2|M|^2 (\langle\alpha|\hat{\Pi}_k(N)|\alpha\rangle + \cos(\varphi)\langle-\alpha|\hat{\Pi}_k(N)|\alpha\rangle). \tag{8}$$

According to the above formulae, we calculated the photon number distribution for each state based on 4, 16, 64, and 256 on/off detectors respectively, and the results are shown in Fig. 2, which verify what we will illustrate in the last section.

3. System similarity and fidelity

To compare the simulated photon number distributions with the ideal ones, we introduced a computable parameter, fidelity,[25] which is expressed as

$$F = \sum_k \sqrt{p_k p_k'}, \tag{9}$$

where p_k' is the probability of the ideal photon number distribution and p_k is the probability we calculated above. Equation (9) can also be understood as the similarity of two classical probability distributions. Assuming all detectors are perfect, we first calculated the fidelities of six quantum states with different numbers of on/off detectors, and the results are presented in Fig. 3(a). It can be clearly observed that as the number of detectors increases, the fidelity tends to be unity. For squeeze states, their photon number has a higher probability of 1 or 2, and the fidelity is fluctuant when less detectors are used.

Figure 3(b) shows the variation of the fidelity as a function of quantum efficiency for the six quantum states. We find that with the quantum efficiency increasing, the fidelities of all the quantum states increase. In particular, the fidelities of the coherent and thermal states, which are close to the ideal states, change slowly, while the fidelities of the other four states change sharply.

This phenomenon is mainly due to the original photon number distributions of these quantum states. States that have non-continuous distributions, such as Fock, squeeze-vacuum, and odd coherent states, have intervals in their photon number distributions. After being detected by the on/off detectors, higher photon number distributions will partly transfer to lower ones. Therefore, the probability of the original photon number states is lower, while that of the adjacent photon number states is higher, resulting in a sharp change in fidelity. On the contrary, for states that have a continuous distribution, both the higher and lower photon number distributions will partly transfer to a lower photon number distribution. As a result, each photon number state will change by a small amount. In other words, the corresponding fidelity will change slowly. The processes are illustrated in Fig. 4.

Figure 3(c) shows the variation of the fidelity as a function of dark counts. We find that all fidelities decrease as the dark counts increase except for the thermal state. Since dark counts are generated from thermal noise, it can compensate the same loss of photon counts due to finite detectors.

Figure 3(d) shows the variation of the fidelities of the Fock states as a function of the mean photon number and the number of detectors for $\nu = 500\ \mathrm{s}^{-1} \times 10\ \mathrm{ns} = 5\times10^{-6}$ (dark count ν is the product of a single photon counter's response time and dark counts per second) and $\eta = 0.9$. These parameters can easily be obtained in experiments. Besides the

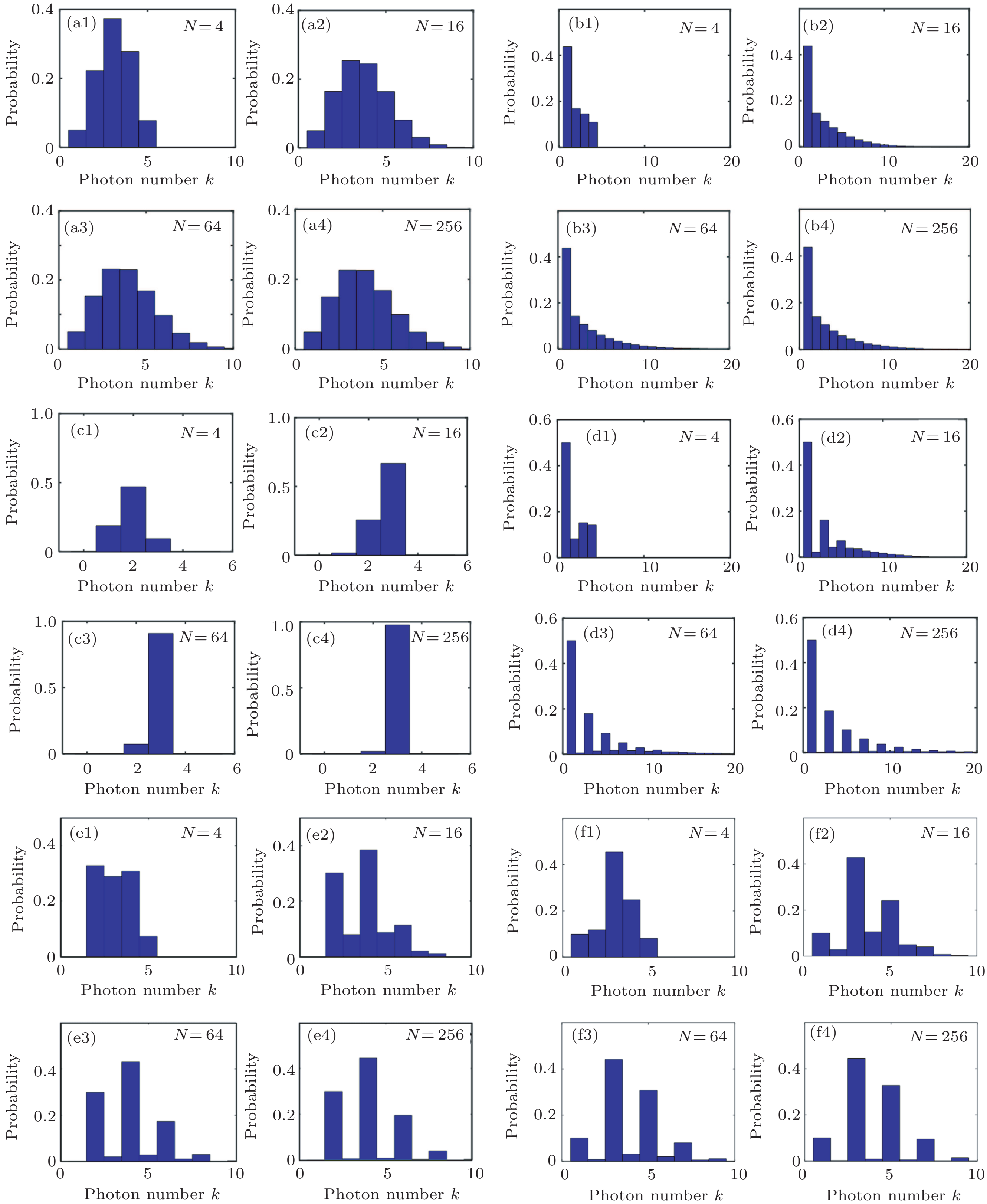

Fig. 2. (color online) Six quantum states with a mean photon number of 3 detected by on/off detector arrays. Photon number distributions of (a) coherent, (b) thermal, (c) Fock, (d) squeeze-vacuum, (e) even coherent, and (f) odd coherent states with $N = 4$, 16, 64, and 256, respectively.

information we obtained in Fig. 3(a), we also find that all fidelities decrease as the mean photon number increases. This phenomenon is mainly due to the higher probability for more than one photon entering in the same port.

4. System diversity and relative entropy

In thermodynamics,[26] entropy is often used to characterize the degree of disorder of a system. In quantum systems,

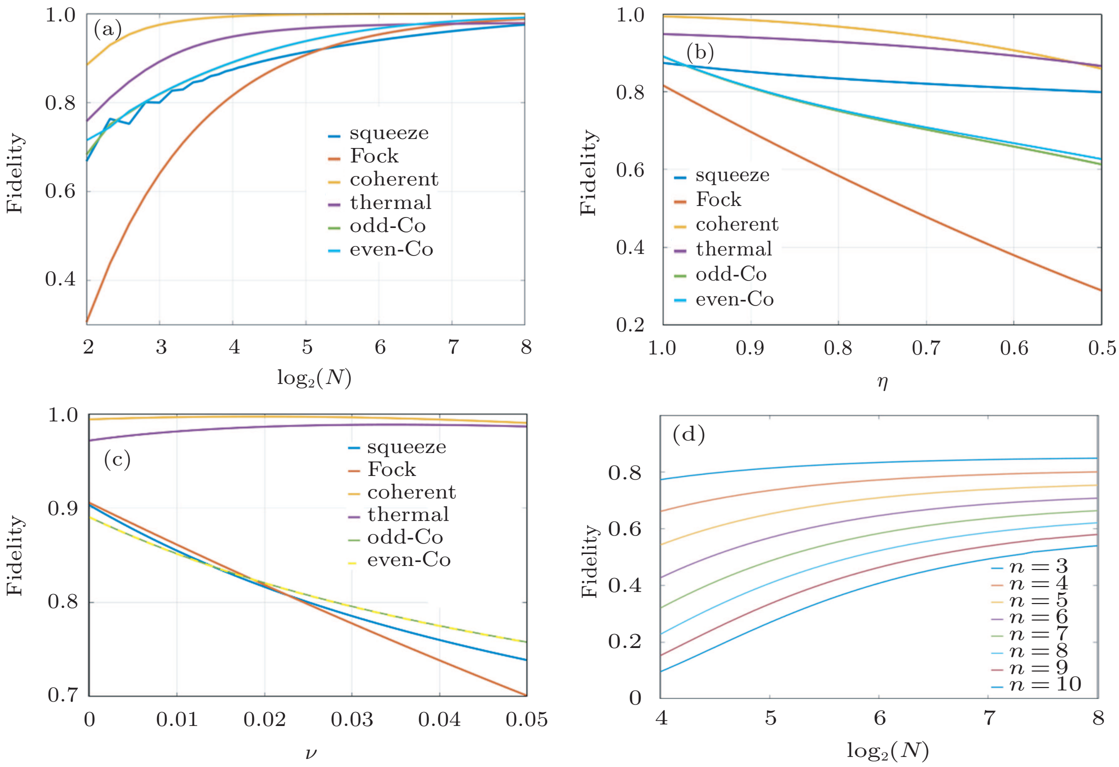

Fig. 3. Six quantum states with a mean photon number of 3 detected by on/off detector arrays. (a) Calculated fidelities of each state based on perfect on/off detector arrays. (b) and (c) Calculated fidelities of each state based on 16 on/off detectors with finite quantum efficiency and dark counts. (d) Calculated fidelities of Fock state based on imperfect on/off detector arrays for $\nu = 5 \times 10^{-6}$ and $\eta = 0.9$.

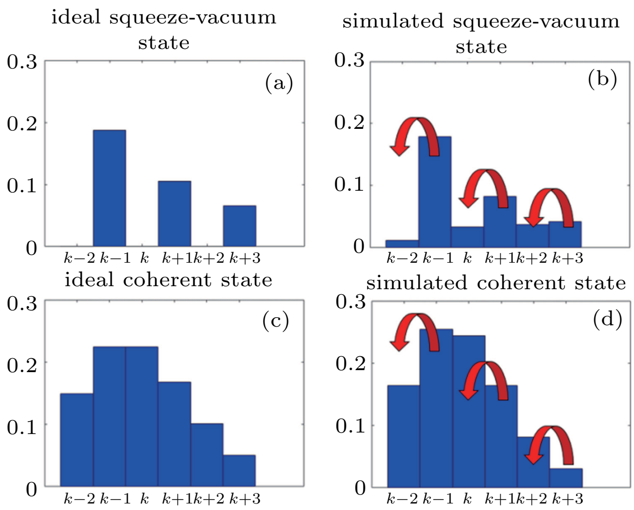

Fig. 4. Photon number distributions of both ideal and simulated ones for (a) and (b) calculated squeeze-vacuum state, (c) and (d) calculated coherent state.

we can also use relative entropy to describe photon number distributions, which can be written as

$$E = \sum_{k=0}^{N} p'_k \log_2 \frac{p'_k}{p_k}. \tag{10}$$

Information entropy[27] is known to indicate the degree of ordering of a system.

It is quite different from the fidelity in the sense that the more ordered a system is, the higher the information entropy will be. Therefore, the relative entropy[28] can describe the distance between two probability distributions in some degree, which, in other words, can also be interpreted as the system diversity. A lower relative entropy here indicates that the simulated photon number distributions are approaching to the ideal distribution, in contrast to the fidelities. Figure 5(a) plots the calculated relative entropy of the six quantum states as a function of the number of on/off detectors. Like the fidelity, the relative entropy can also be used to distinguish between the photon number distributions simulated from the on/off detector array and the ideal distribution. We calculated the relative entropy with the increasing noise. The results are presented in Figs. 5(b) and 5(c), which show similar phenomena to the fidelities we illustrated in Fig. 4. We also calculated the variation of the relative entropy of the Fock states as a function of the mean photon number and the number of detectors for $\nu = 5 \times 10^{-6}$ and $\eta = 0.9$, as shown in Fig. 5(d). It shows the same information which we have obtained in Fig. 3(d). In addition, the relative entropy can be used not only in quantum information, but also in classical information. In conclusion, the relative entropy may be more useful in future photon counting statistics.

5. Conclusion

We have derived a series of universal photon counting formulae for six common quantum states based on multiple on/off detectors. The formulae can be applied to obtain ideal and simulated photon number distributions. We have simulated the photon number distributions and found that increasing

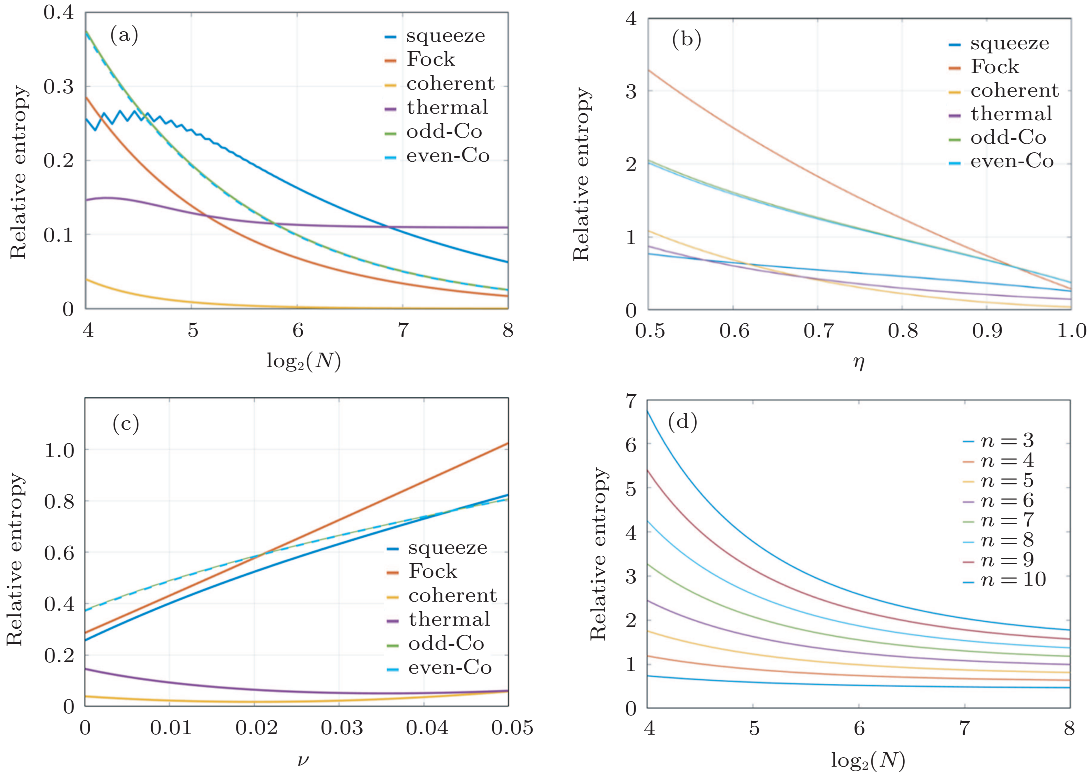

Fig. 5. Six quantum states with a mean photon number of 3 detected by on/off detector arrays. (a) Calculated relative entropy for the six states as a function of detector number. (b) and (c) Calculated relative entropy of each state based on 16 on/off detectors with finite quantum efficiency and dark counts. (d) Calculated relative entropy of the Fock state based on imperfect on/off detector arrays for $\nu = 5\times10^{-6}$ and $\eta = 0.9$.

the number of detectors can result in a higher performance in both perfect and imperfect detections. We then derived and calculated the fidelity of the simulated photon counting distributions, which can assist us in choosing a suitable number of detectors to obtain a near-perfect photon number distribution of each state. We have also derived and calculated the relative entropy of the obtained photon counting distributions, which exhibit an opposite tendency to that of the fidelity. We found that for states with continuous distributions, the detection performance is better than that of the states with non-continuous distributions. Therefore, we should choose near-perfect detectors for non-continuous state detections, while imperfect detectors are acceptable for continuous state detections.

Appendix A

The photon detection probability equations can be derived by the following process from Eq. (2):

$$
\begin{aligned}
&\hat{\Pi}_k(N)\\
&= : C_N^k \exp\left[\frac{-(\eta\hat{n}+N\nu)}{N}\right]^{N-k}\left(I-\exp\left[\frac{(\eta\hat{n}+N\nu)}{N}\right]\right)^k :\\
&= : C_N^k \exp\left[\frac{-(\eta\hat{n}+N\nu)}{N}\right]^{N}\exp\left[\frac{-(\eta\hat{n}+N\nu)}{N}\right]^{-k}\\
&\quad\times\left(I-\exp\left[\frac{(\eta\hat{n}+N\nu)}{N}\right]\right)^k :\\
&= : C_N^k \exp\left[-(\eta\hat{n}+N\nu)\right]\left(\exp\left[\frac{(\eta\hat{n}+N\nu)}{N}\right]-I\right)^k :\\
&= : C_N^k \exp\left[-(\eta\hat{n}+N\nu)\right]\sum_{j=0}^{k} C_k^j \exp\left[\frac{j(\eta\hat{n}+N\nu)}{N}\right](-1)^{k-j} :\\
&= \sum_{j=0}^{k} C_N^k C_k^j (-1)^{k-j} : \exp\left[-(\eta\hat{n}+N\nu)\left(1-\frac{j}{N}\right)\right] :. \qquad \text{(A1)}
\end{aligned}
$$

Considering that $:\exp(-\lambda\hat{n}):=\sum(1-\lambda)^n|n\rangle\langle n|$, where $\lambda = \eta(1-j/N)$, we obtain

$$
\begin{aligned}
\hat{\Pi}_k(N) =& \sum_{n=0}^{\infty} C_N^k\left\{\sum_{j=0}^{k} C_k^j(-1)^{k-j}\exp\left[-N\nu\left(1-\frac{j}{N}\right)\right]\right.\\
&\left.\times\left[1-\eta\left(1-\frac{j}{N}\right)\right]^n\right\}|n\rangle\langle n|. \qquad \text{(A2)}
\end{aligned}
$$

Combining Eqs. (1) and (A2), we have

$$
\begin{aligned}
P_k(\psi) =& \sum_{n=0}^{\infty} C_N^k\left\{\sum_{j=0}^{k} C_k^j(-1)^{k-j}\exp\left[-N\nu\left(1-\frac{j}{N}\right)\right]\right.\\
&\left.\times\left[1-\eta\left(1-\frac{j}{N}\right)\right]^n\right\}\cdot|\langle n|\psi\rangle|^2\\
=& \exp(-N\nu)\sum_{n=0}^{\infty}\sum_{j=0}^{k} C_N^k C_k^j(-1)^{k-j}\cdot\sum_{m=0}^{\infty}\frac{(j\nu)^m}{m!}\\
&\times\sum_{l=0}^{n} C_n^l(1-\eta)^{n-l}\left(\frac{\eta j}{N}\right)^l|\langle n|\psi\rangle|^2\\
=& \sum_{n=0}^{\infty}\sum_{l=0}^{n} C_n^l(1-\eta)^{n-l}\eta^l|\langle n|\psi\rangle|^2\sum_{m=0}^{\infty}\frac{\nu^m}{m!}
\end{aligned}
$$

$$\times\ \exp(-Nv)N^{-l}C_N^k\cdot\sum_{j=0}^{k}C_k^j(-1)^{k-j}j^{m+l}. \quad \text{(A3)}$$

Considering that

$$S(m+l,l)=\frac{1}{k!}\sum_{j=0}^{k}C_k^j(-1)^{k-j}j^{m+l},$$

we obtain

$$\begin{aligned}P_k(\psi)\ =&\ \sum_{l=0}^{\infty}\sum_{n=l}^{\infty}C_n^l(1-\eta)^{n-l}\eta^l|\langle n|\psi\rangle|^2\sum_{m=0}^{\infty}\frac{v^m}{m!}\\ &\times\ \exp(-Nv)N^{-l}S(m+l,l)P_N^k. \end{aligned}\quad \text{(A4)}$$

Next, we normalize the equation by the $\sum P_k(\psi)=1$ relation.

First, by using the $\sum S(n,l)P_m^k=m^n$ relationship, we obtain

$$\begin{aligned}\sum_{k=0}^{N}M_l(N,k,v)\ =&\ \sum_{m=0}^{\infty}\frac{v^m}{m!}\exp(-Nv)\sum_{k=0}^{N}N^{-l}S(m+l,l)P_N^k\\ =&\ \sum_{m=0}^{\infty}\frac{(Nv)^m}{m!}\exp(-Nv)\\ =&\ 1. \end{aligned}\quad \text{(A5)}$$

Combining Eqs. (A4) and (A5), we obtain

$$\begin{aligned}\sum_{k=0}^{N}P_k(\psi)\ =&\ \sum_{l=0}^{\infty}\sum_{n=l}^{\infty}C_n^l(1-\eta)^{n-l}\eta^l|\langle n|\psi\rangle|^2\\ =&\ \sum_{n=0}^{\infty}\sum_{l=0}^{n}C_n^l(1-\eta)^{n-l}\eta^l|\langle n|\psi\rangle|^2\\ =&\ \sum_{n=0}^{\infty}|\langle n|\psi\rangle|^2\\ =&\ 1. \end{aligned}\quad \text{(A6)}$$

Appendix B

The photon detection probability equations of the squeezed-vacuum states can be derived in the following process. By substituting the squeezed-vacuum wave function into Eqs. (3(a)) and (3), we obtain

$$\begin{aligned}P_k(\xi)\ =&\ C_N^k\sum_{j=0}^{k}C_k^j(-1)^{k-j}\exp[-v(1-j/N)]\\ &\cdot\left\{\mathrm{ch}^2\xi-[1-\eta(1-j/N)]^2\sinh^2\xi\right\}^{-1/2}. \end{aligned}\quad \text{(B1)}$$

The last term is

$$\begin{aligned}&\left\{\cosh^2\xi-\left[1-\eta\left(1-\frac{j}{N}\right)\right]^2\sinh^2\xi\right\}^{-1/2}\\ &=A\left\{1-2xy\left(\frac{j}{N}\right)-y^2\left(\frac{j}{N}\right)^2\right\}^{-1/2}\\ &=A\sum_{l=0}^{\infty}P_l(\mathrm{i}x)(-\mathrm{i}y)^l(j/N)^l, \end{aligned}\quad \text{(B2)}$$

where the terms A, x, and y are defined in the main text.

Similar to Appendix A, we obtain

$$\begin{aligned}P_k(\xi)\ =&\ A\sum_{l=0}^{\infty}P_l(\mathrm{i}x)(-\mathrm{i}y)^l\\ &\cdot\sum_{m=0}^{\infty}\frac{1}{m!}v^m\exp(-v)\frac{1}{N^{l+m}}S(l+m,k)P_N^k. \end{aligned}\quad \text{(B3)}$$

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